**Title: Regular Pentagons, “Star Polygons”, and The Golden Ratio**

**Brief Overview:**

This learning unit is designed for students to explore the golden ratio, golden rectangles, and find the golden ratio in regular pentagons. Students will also have the opportunity to use Geometer’s Sketchpad to rotate, translate, and dilate various points and line segments as well as solving problems.

**NCTM 2002 Principles for School Mathematics:**

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*
- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*
- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*
- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*
- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*
- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students’ learning.*

**Links to NCTM 2002 Standards:**

- **Content Standards**

**<u>Number and Operations</u>**
Students will understand the relationships between various segment lengths. They will calculate the area of given polygons, compute the ratios of various quantities and solve proportions.

**<u>Algebra</u>**
Students will compare and analyze mathematical situations by using the golden ratio.

**Geometry**

Students will analyze the characteristics and properties of polygons. Particular attention is paid to rectangles and pentagons. They will find relationships between geometric shapes and develop mathematical arguments to verify geometric figures using coordinate plane and other representation systems

**Measurement**

Students will be able to understand measurable attributes of objects and the units, systems and the process of measurement. They will apply appropriate techniques and tools to determine measurements.

- **Process Standards**

**Mathematics as Problem Solving, Reasoning and Proof, Communication, Connections and Representation**

These five process standards are integrated throughout this learning unit. They emphasize the need to help students develop the processes that are the major means for doing mathematics, thinking about mathematics, understanding mathematics, and communicating mathematics.

The students will use Geometer's Sketchpad to demonstrate geometric relationships and to enhance technology in the learning unit.

**Links to Maryland High School Mathematics Core Learning Units:**

**Geometry, Measurement, and Reasoning**

- **2.1.1**
  The student will analyze the properties of geometric figures and will construct geometric figures using technology.

- **2.1.2**
  The student will identify and verify properties of geometric figures using the co-ordinate plane and concepts from algebra.

- **2.1.3**
  The student will use transformations to move figures, create designs, and demonstrate geometric properties.

- **2.1.4**
  The students will construct and validate properties of geometric figures using appropriate tools and technology.

- **2.2.1**
  The students will identify and verify congruent and/or similar figures and apply equality or proportionality of their corresponding parts.

- **2.2.2**
  The students will solve problems using two-dimensional figures.

- **2.2.3**
  The student will use inductive or deductive reasoning.

- **2.3.1**
  The students will use algebraic and geometric properties to measure indirectly.

- **2.3.2**
  The student will use techniques of measurement and will calculate various lengths. And will compare area of two dimensional figures and their parts.

**Grade/Level:**

Grades 9 – 12, Geometry

**Duration/Length:**

Two to three class periods, approximately 45-50 minutes in length

**Prerequisite Knowledge:**

Students should have working knowledge of the following skills:

- Applying basic geometric figures
- Using Geometer's Sketchpad to draw basic geometric figures
- Using Geometer's Sketchpad to calculate
- Applying similar polygons and proportions
- Applying Pythagorean Theorem

**Student Outcomes:**

Students will:

- Use the basic features of the Geometer's Sketchpad.
- Use the Geometer's Sketchpad to demonstrate rotations, translations and dilations of geometric figures.
- Describe the golden ratio in terms of how it appears in art and architectural designs.
- Identify what segments in a star polygon are related by the golden ratio.
- Calculate and compare areas of given geometric figures.

**Materials/Resources/Printed Materials:**

- Textbook
- Computer with Geometer's Sketchpad
- Student Worksheets, Homework Sheets, and Assessment Sheets
- Answer Sheets for Worksheets, Homework, and Assessment
- Teacher's notes

**Development/Procedures:**

This is an activity-based unit with several worksheets that students will complete and have checked. The students will need at least two days to do this activity. These are the suggested procedures for each day:

***Day 1***

(1) Teacher will review any necessary computer skills procedures. (Worksheets and Assessment were developed using Geometer's Sketchpad 3.0.)
(2) Teacher should lead a class discussion on the golden ratio, with examples.
(3) Student will complete Worksheet 1 using Geometer's Sketchpad.
(4) Depending on time and students' ability, worksheet 2 can be used. (*Completion of Worksheet 2 is not a prerequisite for worksheet 3).*
(5) Homework assignment will be given to review and assess students' achievement.

***Day 2***

(1) Teacher should review the previously learned materials.
(2) Have students complete Worksheet 3 using Geometer's Sketchpad.
(3) Students will complete an assessment to conclude the unit.

**Assessment:**

Students will use Geometer's Sketchpad to create the constructions and solve problems about golden rectangles and the golden ratio. The use of Geometer's Sketchpad will enable visual and hands-on assessment. A formal assessment is included at the end of the unit with a scoring guide provided.

**Extension/Follow Up:**

The students can use Microsoft Excel to perform an investigation related to the Fibonacci sequence.

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**References:**

Gullberg, Jan. Mathematics: From The Birth Of Numbers 1997. W.W Norton & Co., New York.

## Teacher Notes

### About Worksheet 1

- This worksheet introduces the Golden Ratio and the Golden Rectangle. We have included two examples on the worksheet, one from art and one from architecture.
- In order for your students to see as much of the ratio as possible, you may want to have them set the Preferences (in the Display menu) in Sketchpad to thousandths for "Distance Unit Precision" and "Slope and Calculation Precision." If you don't want to deal with this, Step 6 discusses that they may get 1.6 or 1.62 (which is okay).
- The Sketchpad activity begins with the students constructing a segment that demonstrates the Golden Ratio using Dilation. They will use the Calculate feature to show the ratio.
- Using their segment, the students will then construct a Golden Rectangle. The directions require the students to use rotations, mark vectors, and translate segments along vectors. The worksheet takes the students step-by-step through each of these transformations, and prior knowledge of these topics is not necessary.
- Students will complete two examples in which they apply the Golden Ratio to Golden Rectangles.
- **If you are not familiar with any of the techniques needed to carry out the constructions, we strongly suggest that you work through all of the steps prior to the class period.**

### About Worksheet 2 **(Optional)**

- "Where did 1.618 come from anyway?" is **optional** and not required for successful completion of Worksheet 3.
- Prerequisite algebra skills include cross multiplication, rewriting a quadratic in standard form, substituting values into the quadratic formula, and simplifying radicals.
- There is a sample quadratic equation for the students to work through prior to working with the proportion leading to the Golden Ratio.

### About the Homework

- The homework is designed to assess the student's understanding of the activities in Worksheet 1.

## About Worksheet 3

- This worksheet is an investigation of how the Golden Ratio appears in a regular pentagon. Starting with a segment, the students will rotate the segment by 72 degrees 4 times to create the regular pentagon.
- Step 6 includes two questions regarding the significance of the 72 degree rotations. If you have not covered central angles in regular figures, these questions should be omitted.
- Students will then hide the segments and then add lines to the picture to create what we are referring to as a "star polygon." They will then find lengths in the "star polygon", compare them, and find the Golden Ratio.
- Students will then find the area of the "star polygon" (by constructing its interior) and compare it with the area of the pentagon (by constructing its interior). The ratio is **twice the Golden Ratio**. We cannot explain this outcome, and would be happy to hear your ideas.
- **If you are not familiar with any of the techniques needed to carry out the constructions, we strongly suggest that you work through all of the steps prior to the class period.**
- Page 4 of the worksheet is **optional** and requires student knowledge of Microsoft Excel. The activity will show that the limit of the successive quotients of the Fibonacci sequence approaches the Golden Ratio. Successful completion of this activity is not a prerequisite for the assessment.

## Assessment

- The assessment is designed for a 40-50 minute class period and focuses on the activities in Worksheets 1 and 3.

**Worksheet 1**

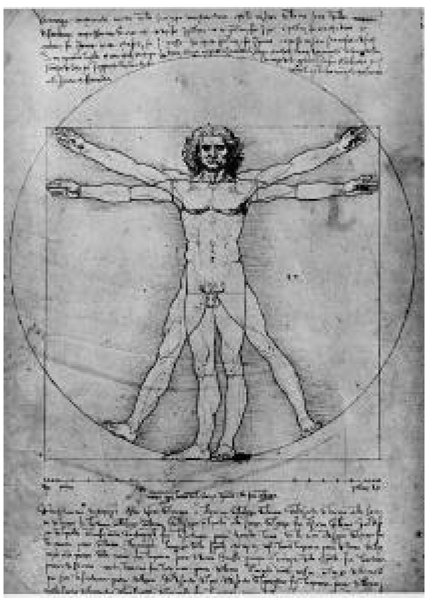
**NAME ________________________________ DATE _______________________**

**The Golden Ratio and Golden Rectangles**

The Golden Ratio was, as written in "Mathematics: From Birth to Numbers" by Jan Gullberg,

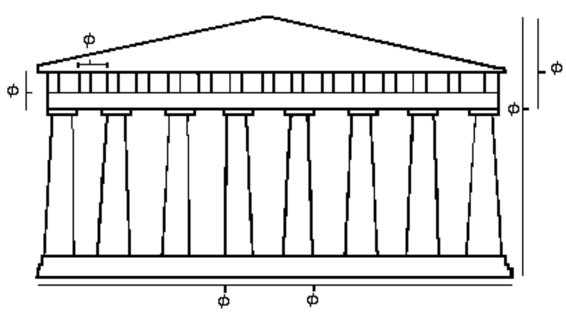
> …for about 2500 years, a recognized aesthetic guide in art, even a decree absolute, governing the shape and disposition of drawings and paintings, and in architecture, where the facades of many public edifices were proportioned according to this ratio." (Pg. 418)

The Golden Ratio is demonstrated in the dimensions of the following examples:



da Vinci's *Vitruvian man*

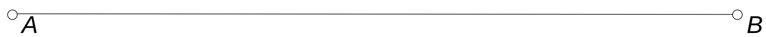
Da Vinci was fascinated with the proportions in the human body. In this picture, the distance from the navel to the top of his head, divided by the distance from the bottom of his feet to the navel, is the Golden Ratio.



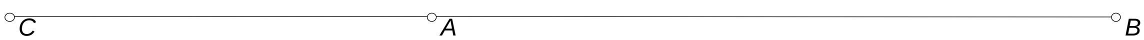

The Golden Ratio (as represented by the Greek letter phi, Φ) appears in many of the dimensions and friezes of The Parthenon.

You are now going to use Geometer’s Sketchpad to investigate some of the properties of The Golden Ratio.

1. In Geometer’s Sketchpad, open a new sketch and create a segment *AB*.




2. Double-click Point *B* (you will get a “bulls-eye” effect). With Point *B* highlighted, select Point *A* and the segment.
3. Go to the Transform menu and select Dilate. In the Scale Factor Box, type in 1.618 for the (New) value, and 1.000 for the (Old) Value. Click OK.
4. You may need to move your new segment in order to see the entire figure. Label the new endpoint *C*.




5. Select the endpoints of segment *AB*, go the Measure menu, and choose Distance.
6. Select the endpoints of segment *CA*, go the Measure menu, and choose Distance.
7. Go to the Measure menu and choose Calculate. Click on the measure of *AB*, click on the division symbol, and then click on the measure of *CA*. Your answer should be 1.618. If your calculation is 1.6 or 1.62, it’s okay. This is an approximation of the **Golden** Ratio. The lengths of this segment also satisfy the following proportion:

$$\frac{AB}{AC} = \frac{BC}{AB}$$

8. Select and drag Point *B*. What is changing? What is remaining constant?

Now you are going to build a Golden Rectangle.

1. To do this you will be creating a rectangle whose length is *CB* and whose width is *AC*.
2. Using segment *CB*, select Points *C* and *A*. Go to the Construct menu and choose Segment.
3. Double click *A*. With A highlighted, select segment *CA* and Point *C*. Go to the Transform menu and choose Rotate. In the Rotate box, type in -90. Click OK. Label the new endpoint *D*.
4. Select *A* and then *B*. Go to the Transform menu and choose Mark Vector “*A*->*B*”. You may see a cool visual effect, but it’s quick. If you missed it, do this step again.
5. Select segment *AD* and Point *D*. Go to the Transform menu and choose Translate. Click OK. Label the new point *E*.

6. Select Points *D* and *E*.  Go to the Construct menu and choose segment.  You have now created a **Golden Rectangle**.  Check this by finding the ratio of *AB* to *AD*.
7. Click and drag on Point *B*.  What is changing?  What remains constant?

_______________________________________________

_______________________________________________

8. As you change the dimensions of your rectangle, its dimensions are "visually appealing" (according to the Golden Ratio) and would serve well as a façade for a building or other rectangular shape.

Now you will apply the Golden Ratio to find the missing lengths in some Golden Rectangles.  Use 1.618 for the Golden Ratio and round your answers to the nearest tenth.

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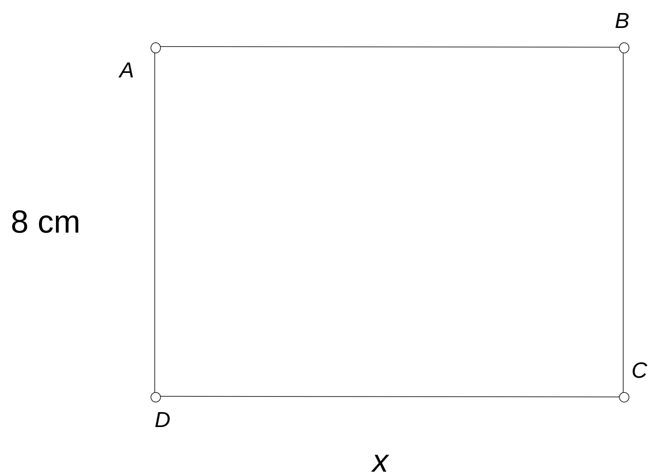
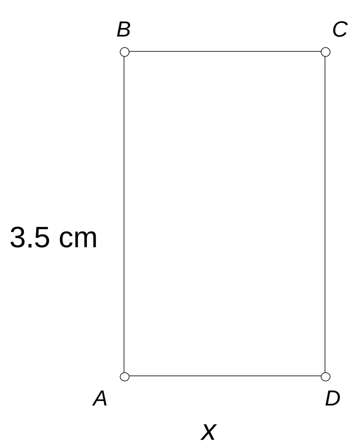



2.



**Worksheet 1 – Teacher's Guide**

**NAME ___________________________________ DATE __________________________**

## The Golden Ratio and Golden Rectangles

The Golden Ratio was, as written in "Mathematics: From Birth to Numbers" by Jan Gullberg,

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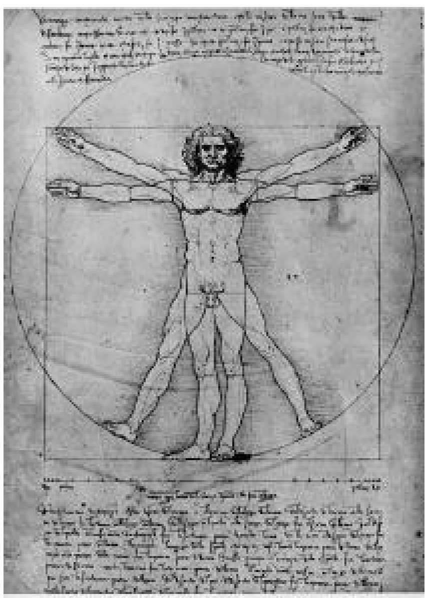
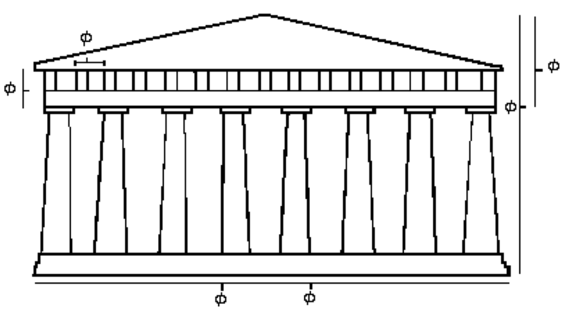


da Vinci's *Vitruvian man*

Da Vinci was fascinated with the proportions in the human body. In this picture, the distance from the navel to the top of his head, divided by the distance from the bottom of his feet to the navel, is the Golden Ratio.



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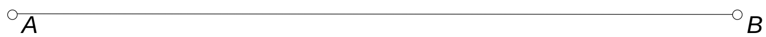


2. Double-click Point *B* (you will get a "bulls-eye" effect). With Point *B* highlighted, select Point *A* and the segment.
3. Go to the Transform menu and select Dilate. In the Scale Factor Box, type in 1.618 for the (New) value, and 1.000 for the (Old) Value. Click OK.
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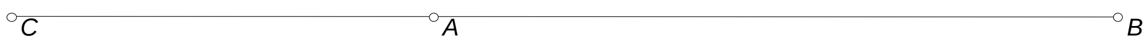


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8. Select and drag Point *B*. What is changing? What is remaining constant?
   **The segment lengths are changing; the ratio is constant.**

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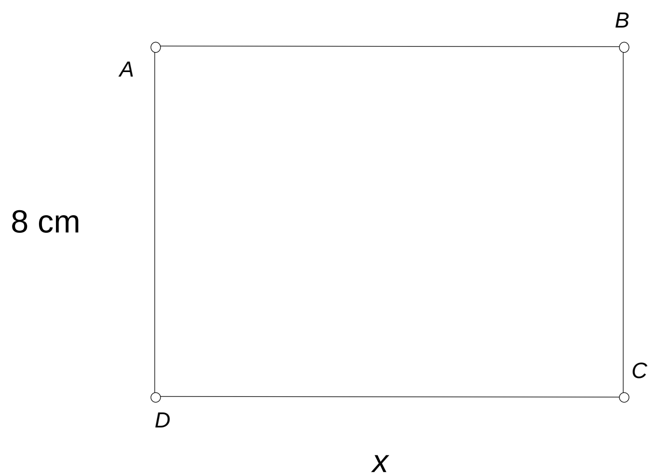
7. Click and drag on Point *B*. What is changing? What remains constant?
   **The segment lengths are changing; the ratio is constant.**

8. As you change the dimensions of your rectangle, its dimensions are "visually appealing" (according to the Golden Ratio) and would serve well as a façade for a building or other rectangular shape.

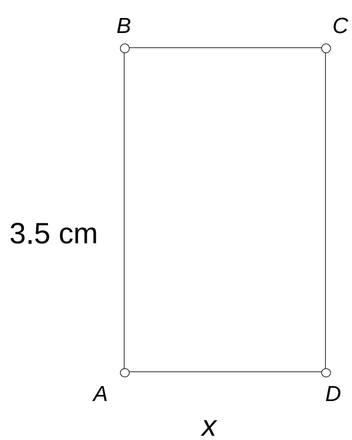
Now you will apply the Golden Ratio to find the missing lengths in some Golden Rectangles. Use 1.618 for the Golden Ratio and round your answers to the nearest tenth.

1.




***x* = 12.9**

2.

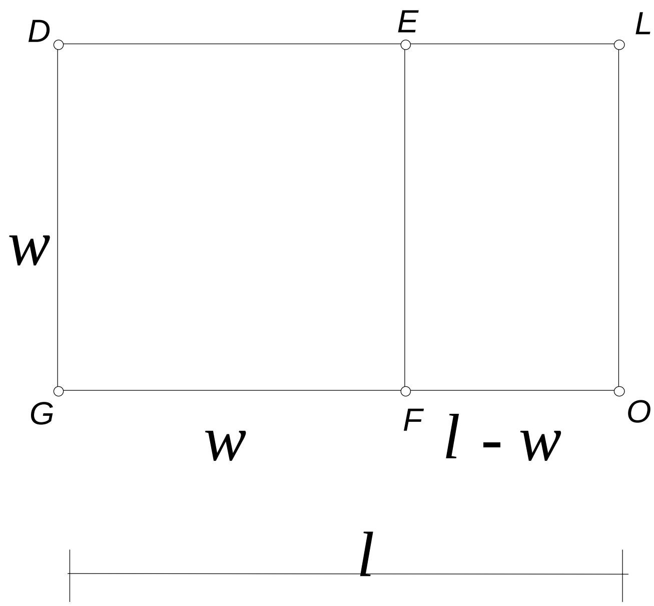



***x* = 2.2**

**Worksheet 2**

**NAME ___________________________________   DATE _________________________**

**Where did 1.618 come from anyway?**




If a square (*GFED*) is formed at one end of a golden rectangle (*GOLD*), then the remaining piece (*FOLE*) is similar to the original rectangle. This relationship leads to the following proportion:

$$\frac{l}{w} = \frac{w}{l - w}$$

The ratio of the longer side *(l)* of Rectangle *GOLD* to the shorter side (*w*) of Rectangle *GOLD* (*l/w)* is the golden ratio. To find the value, we must work with the proportion. Since the width of the original rectangle (*GOLD*) can be any number, we will set the width equal to 1. Now our proportion is in one variable.

$$\frac{l}{1} = \frac{1}{l - 1}$$

1. Cross multiply the proportion.

2. You now have a quadratic equation. One way to solve a quadratic equation is to use the Quadratic Formula. In order to use the formula, the equation must be in standard form ( $ax^2+bx+c=0$ ). For example, suppose you were working with the equation $x^2-5x=6$. In order to write the equation in standard form, you would need to add 6 to both sides, giving you $x^2-5x-6=0$. Now, rewrite your equation from Step 1 in standard form.

3. Once the equation is in standard form ( $ax^2+bx+c=0$ ), the quadratic formula uses three values: the coefficient of the quadratic term (*a*), the coefficient of the linear term (*b*), and the constant term (*c*). In the equation $x^2-5x-6=0$, $a=1$, $b=-5$, and $c=-6$. You may recall that the quadratic formula is:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

   Substituting the values from above gives you:

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-6)}}{2(1)}=\frac{5\pm\sqrt{25+24}}{2}=\frac{5\pm\sqrt{49}}{2}=\frac{5\pm 7}{2}$$

$$\frac{5\pm 7}{2}=\frac{5+7}{2}\ or\ \frac{5-7}{2}=\frac{12}{2}\ or\ \frac{-2}{2}=6\ or\ -1$$

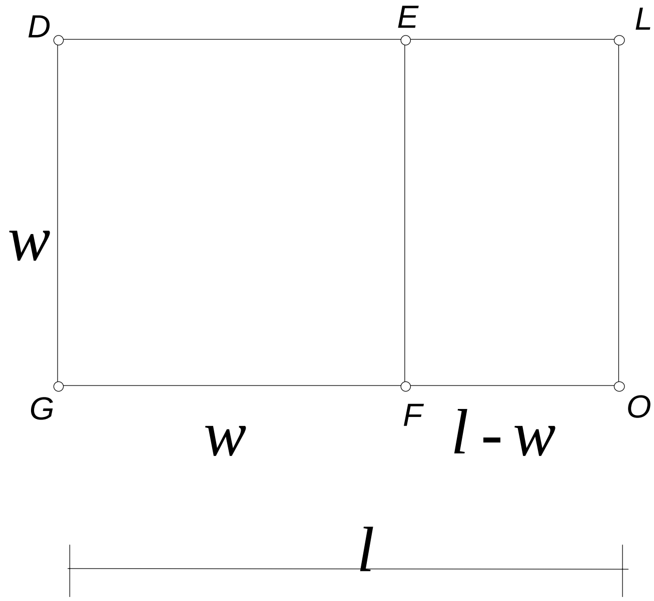
   Using this example as a guide, solve the equation you obtained in Step 2 and round your answers to the nearest thousandth.

We will disregard the negative value of *l*. Earlier we defined the Golden Ratio as $\frac{l}{w}$. Because $l=1.618$ and $w=1$, then $\frac{l}{w}=1.618$. Now you see why the approximation of the Golden Ratio is 1.618.

**Worksheet 2 – Teachers Guide**

**NAME ___________________________________ DATE _________________________**

**Where did 1.618 come from anyway?**




If a square (*GFED*) is formed at one end of a golden rectangle (*GOLD*), then the remaining piece (*FOLE*) is similar to the original rectangle. This relationship leads to the following proportion:

$$\frac{l}{w} = \frac{w}{l-w}$$

The ratio of the longer side *(l)* of Rectangle *GOLD* to the shorter side (*w*) of Rectangle *GOLD* (*l/w)* is the golden ratio. To find the value, we must work with the proportion. Since the width of the original rectangle (*GOLD*) can be any number, we will set the width equal to 1. Now our proportion is in one variable.

$$\frac{l}{1} = \frac{1}{l-1}$$

1. Cross multiply the proportion.

   $\boldsymbol{l^2 - l = 1}$

2. You now have a quadratic equation. One way to solve a quadratic equation is to use the Quadratic Formula. In order to use the formula, the equation must be in standard form ( $ax^2 + bx + c = 0$ ). For example, suppose you were working with the equation $x^2 - 5x = 6$. In order to write the equation in standard form, you would need to add 6 to both sides, giving you $x^2 - 5x - 6 = 0$. Now, rewrite your equation from Step 1 in standard form.

   $\boldsymbol{l^2 - l - 1 = 0}$

3. Once the equation is in standard form ( $ax^2 + bx + c = 0$ ), the quadratic formula uses three values: the coefficient of the quadratic term (*a*), the coefficient of the linear term (*b*), and the constant term (*c*). In the equation $x^2 - 5x - 6 = 0$, $a = 1$, $b = -5$, and $c = -6$. You may recall that the quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values from above gives you:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm \sqrt{25 + 24}}{2} = \frac{5 \pm \sqrt{49}}{2} = \frac{5 \pm 7}{2}$$

$$\frac{5 \pm 7}{2} = \frac{5 + 7}{2} \ or \ \frac{5 - 7}{2} = \frac{12}{2} \ or \ \frac{-2}{2} = 6 \ or \ -1$$

Using this example as a guide, solve the equation you obtained in Step 2 and round your answers to the nearest thousandth.

$$\boldsymbol{l = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2} \approx 1.618 \ or \ -0.618}$$

**They will disregard -0.618.**

We will disregard the negative value of *l*. Earlier we defined the Golden Ratio as $\frac{l}{w}$. Because $l = 1.618$ and $w = 1$, then $\frac{l}{w} = 1.618$. Now you see why the approximation of the Golden Ratio is 1.618.

**Homework Sheet**

1) Do an Internet search on the Golden Rectangle to find three additional examples of its use in art and architecture.

a) ______________________________________________________________

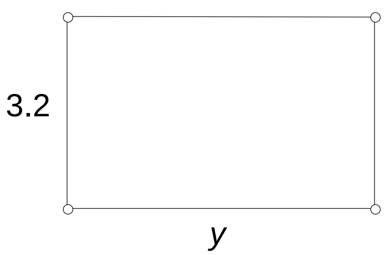
b) ______________________________________________________________

c) ______________________________________________________________

2) Point *A* is positioned such that segments *CA* and *AB* are in the Golden Ratio. Find the value of *x*. Round to the nearest hundredth.

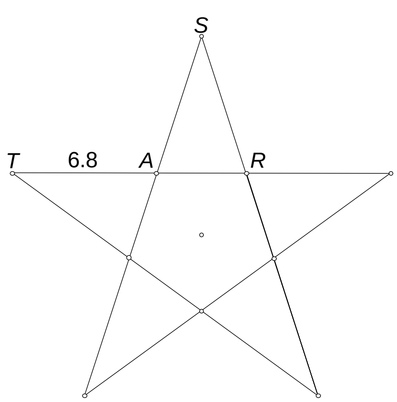

$x$ = _______

3) The rectangle below is a Golden Rectangle. Find the value of *y*. Round to the nearest hundredth.




$y$ = _______

4) Find the lengths of the following segments in the star shown below if *TA* = 6.8. The pentagon shown is regular. Round your answers to the nearest tenth.




a) *SR* = ______ b) *AR* = ______ c) *TR* = ______ d) *SA* = ______

5) Consider a sequence of numbers in which the first two terms are each defined to be 1 and successive terms are defined to be the sum of the two prior terms.

$a_1$, $a_2$, $a_3$, $a_4$, $a_5$, …

1, 1, 2, 3, 5, …

This sequence is called the Fibonacci sequence, in honor of the Italian merchant and mathematician Leonardo of Pisa, or Fibonacci (c. 1170-1250).

a) Find the sixth, seventh, eighth, and ninth terms of the sequence: $a_6$, $a_7$, $a_8$, and $a_9$.

$a_6 =$ ________ $a_7 =$ _______ $a_8 =$ _______ $a_9 =$ _______

b) Compute the values of the following ratios.

$\frac{a_6}{a_5} =$ _______ $\frac{a_7}{a_6} =$ _______ $\frac{a_8}{a_7} =$ _______ $\frac{a_9}{a_8} =$ _______

c) Describe how the ratios that you computed in part b) are changing.

_______________________________________________________________________

**Homework Answer Key**

1) Do an Internet search on the Golden Rectangle to find three additional examples of its use in art and architecture. **There are numerous examples. Here are a few:**

a) **Chartes Cathedral in France**

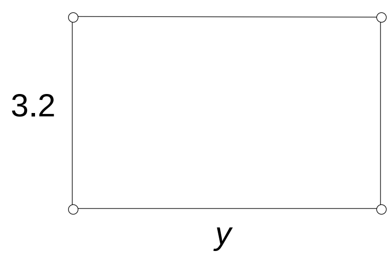
b) **The United Nations building in New York City**

c) **The paintings of Mondrian.**

2) Point *A* is positioned such that segments *CA* and *AB* are in the Golden Ratio. Find the value of *x*. Round to the nearest hundredth.

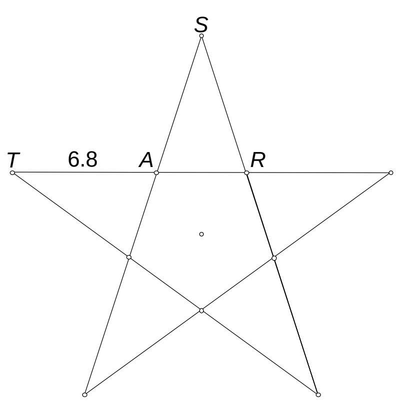

*x* = **3.23**

3) The rectangle below is a Golden Rectangle. Find the value of *y*. Round to the nearest hundredth.




*y* = **5.18**

4) Find the lengths of the following segments in the star shown below if *TA* = 6.8. The pentagon shown is regular. Round your answers to the nearest tenth.




a) *SR* = **6.8** b) *AR* = **4.2** c) *TR* = **11.0** d) *SA* = **11.0**

5) Consider a sequence of numbers in which the first two terms are each defined to be 1 and successive terms are defined to be the sum of the two prior terms.

| | | | | | |
|---|---|---|---|---|---|
| $a_1$, | $a_2$, | $a_3$, | $a_4$, | $a_5$, | … |
| 1, | 1, | 2, | 3, | 5, | … |

This sequence is called the Fibonacci sequence, in honor of the Italian merchant and mathematician Leonardo of Pisa, or Fibonacci (c. 1170-1250).

a) Find the sixth, seventh, eighth, and ninth terms of the sequence: $a_6$, $a_7$, $a_8$, and $a_9$.

$a_6$ = **8** $a_7$ = **13** $a_8$ = **21** $a_9$ = **34**

b) Compute the values of the following ratios. Round to the nearest thousandth.

$\frac{a_6}{a_5}$ = **1.6** $\frac{a_7}{a_6}$ = **1.625** $\frac{a_8}{a_7}$ = **1.615** $\frac{a_9}{a_8}$ = **1.619**

c) Describe how the ratios that you computed in part b) are changing.

**The values are getting closer and closer to the Golden Ratio.**

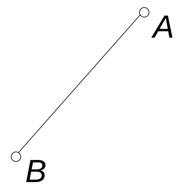
**Worksheet 3**

**NAME ________________________________ DATE _______________________**

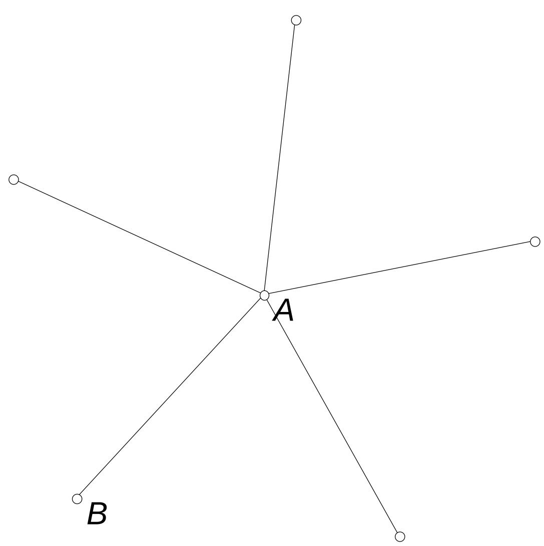
**Using Regular Pentagons to Explore the Golden Ratio**

Using Geometer's Sketchpad you will explore a regular pentagon and how the Golden Ratio appears in different measurements.

1. Open a new sketch and draw a segment *AB* similar to one below.

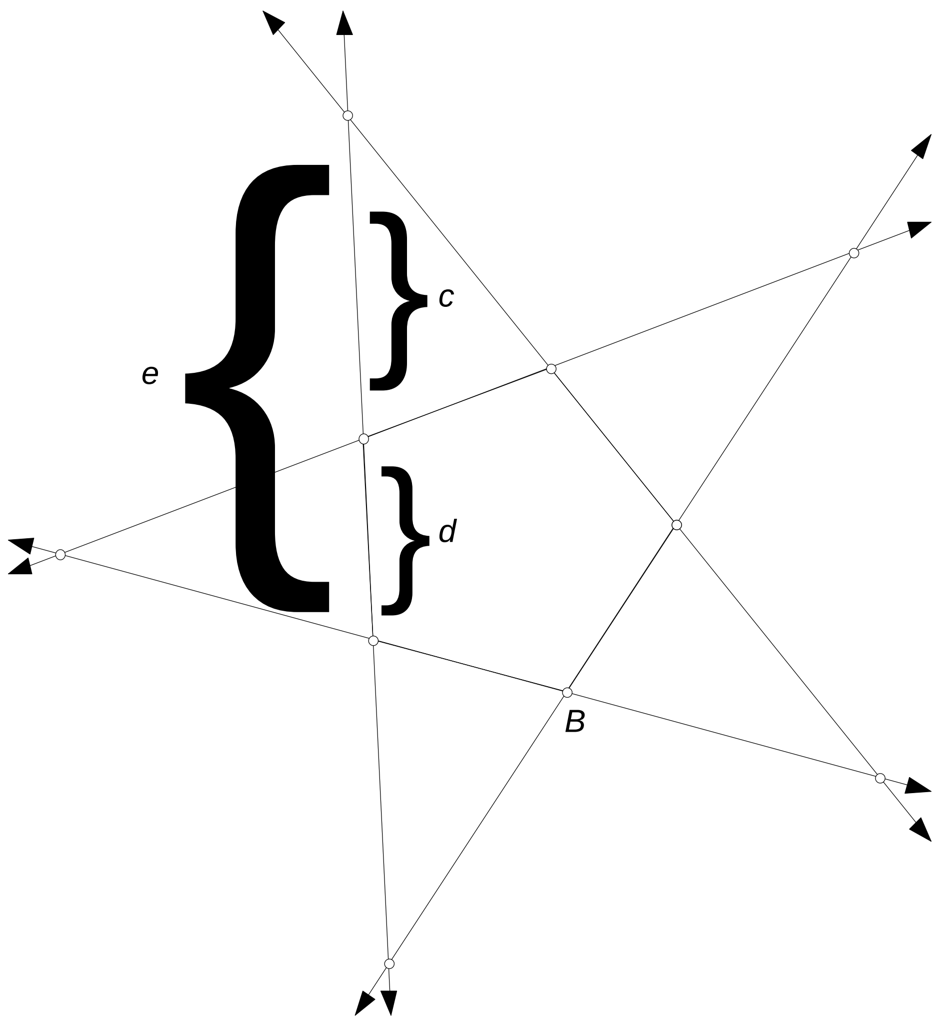



2. Double-click on *A* (you will get the "bulls-eye" effect). With *A* highlighted, select the segment and Point *B.*
3. Go to the Transform menu and choose Rotate. In the Rotate window, type in 72 and click OK.
4. Return to the Transform menu, select Rotate, and press OK.
5. Do this rotation two more times. You should now have a "five-spoked" figure like the one below.




6. Select the 5 endpoints of the figure **in order**. Go to the Construct menu and select Segment. You have just created a regular pentagon. What role did the 72 degree rotations play in the formation of this figure? Can you prove the figure is regular?

7. Select all five "spokes" and point *A*. Go to the Display menu and select Hide Objects.
8. Choose the line icon. Select two consecutive vertices of the pentagon and construct a line (not a segment)
9. Repeat Step 8 for each remaining pairs of consecutive vertices.
10. You have now created a "star polygon" around the pentagon. If necessary, select point *B* and drag it so that the entire star is visible.




Now you will find the ratios of different segments. Using the figure above:

1. Select the endpoints of a segment on your diagram that corresponds to *c*. Measure the length of segment *c*.
2. Select the endpoints of a segment on your diagram that corresponds to *d*. Measure the length of segment *d*.

3. Go to Measure and choose Calculate. Then, click on the length representing segment *c*, click on the division symbol, and then click on the length representing segment *d*. Click OK. Where have you seen this number before?

_______________________________________________________________________

_______________________________________________________________________

4. Grab and drag Point *B*. What is changing? What is remaining constant?

_______________________________________________________________________

_______________________________________________________________________

5. Select the endpoints of a segment on your diagram that corresponds to *e*. Measure the length of segment *e*.
6. Go to Measure and choose Calculate. Then, click on the length representing segment *e*, click on the division symbol, and then click on the length representing segment *c*. Click OK. There it is again!
7. Grab and drag Point *B*. What is changing? What is remaining constant?

_______________________________________________________________________

_______________________________________________________________________

8. Select the vertices of the star polygon **in order**. Go to Construct and select Polygon Interior (this will shade the interior). Click on the interior of the star polygon so it is highlighted, go to Measure, and select Area.
9. Go to Display and choose Hide Polygon.
10. Select the vertices of the pentagon **in order**. Go to Construct and select Polygon Interior. Click on the interior of the pentagon so it is highlighted, go to Measure, and select Area.
11. Go to Measure and select Calculate. Click on the larger area, the division symbol, and then the smaller area. Click OK. How does this number compare with The Golden Ratio? Can you use the Calculate feature to justify your answer?

_______________________________________________________________________

_______________________________________________________________________

12. Grab and drag Point *B*. What is changing? What is remaining constant?

_______________________________________________________________________

_______________________________________________________________________

## Fibonacci numbers and the Golden Ratio

In 1202, Leonardo Fibonacci proposed a problem with an answer that was in the form of the following sequence:

1, 1, 2, 3, 5, 8, 13…

This set of numbers is known as the Fibonacci sequence. As you may notice, starting with the third value, each successive value is the sum of the previous two values. What are the next two values in the sequence?

You are now going to use an Excel spreadsheet to develop the Fibonacci sequence and look at the values of successive quotients.

1. Open an Excel spreadsheet. In cells A1 and A2, enter the number 1. With cell A3 highlighted, go to the formula bar and enter =SUM(A1:A2). Press enter. You should now see 2 in cell A3.
2. Select cell A3 and grab the square in the left hand corner of the cell (you will see a plus sign). Drag down to cell A20 and release. You now have the first twenty values of the Fibonacci sequence. **A quick check:** The value in cell A20 should be 6765.
3. Select cell B3. In the formula bar, enter =(A3/A2). Then, select cell B3 and drag this formula down to cell B20.
4. What do you notice about the values in column B?

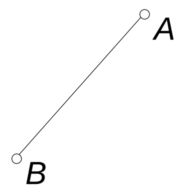
**Worksheet 3 – Teacher's Guide**

**NAME __________________________________ DATE ________________________**

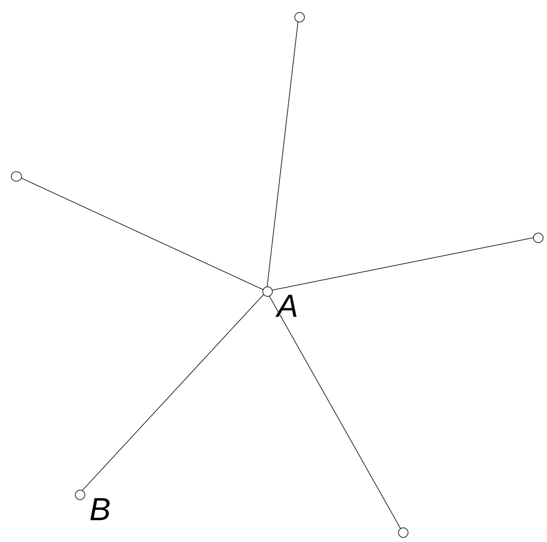
**Using Regular Pentagons to Explore the Golden Ratio**

Using Geometer's Sketchpad you will explore a regular pentagon and how the Golden Ratio appears in different measurements.

1. Open a new sketch and draw a segment *AB* similar to one below.



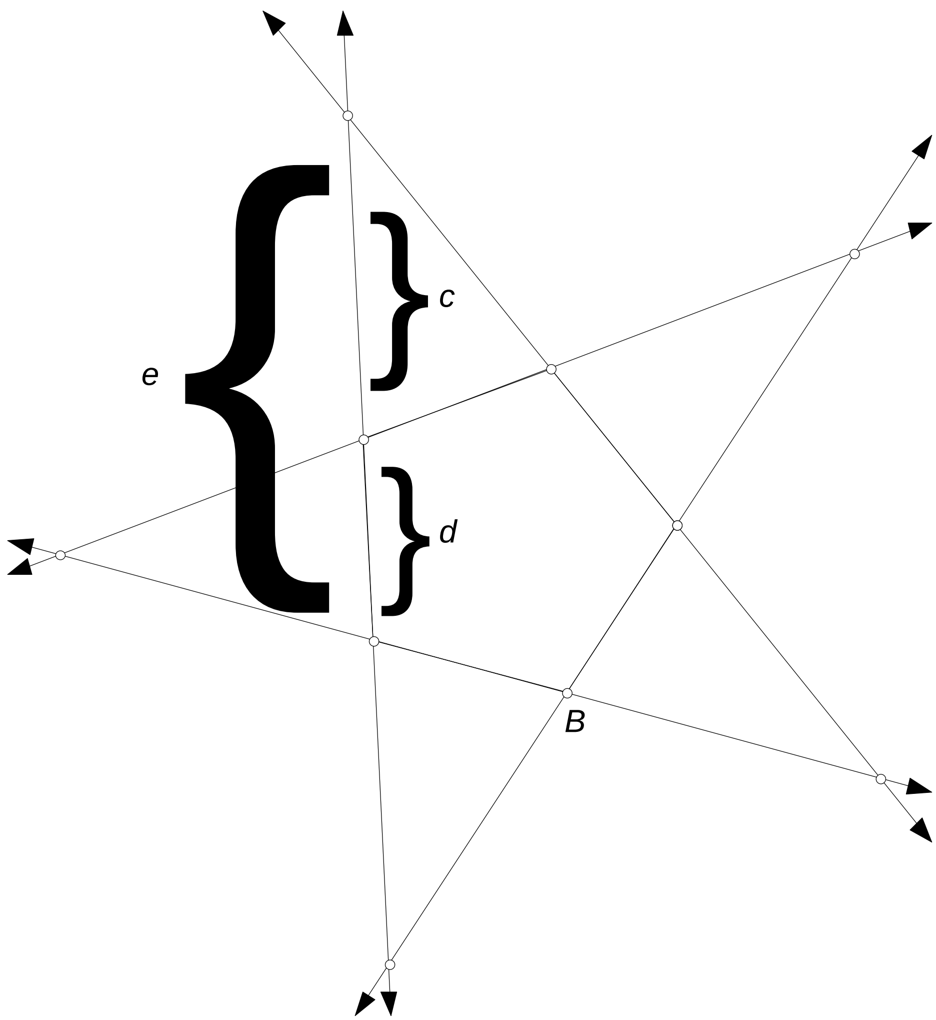

2. Double-click on *A* (you will get the "bulls-eye" effect). With *A* highlighted, select the segment and Point *B*.
3. Go to the Transform menu and choose Rotate. In the Rotate window, type in 72 and click OK.
4. Return to the Transform menu, select Rotate, and press OK.
5. Do this rotation two more times. You should now have a "five-spoked" figure like the one below.




6. Select the 5 endpoints of the figure **in order**. Go to the Construct menu and select Segment. You have just created a regular pentagon. What role did the 72 degree rotations play in the formation of this figure? Can you prove the figure is regular?

**72 degrees is the measure of the central angle. SAS Triangle Congruence and CPCTC can be used to show all five sides are congruent.**

7. Select all five "spokes" and point *A*. Go to the Display menu and select Hide Objects.
8. Choose the line icon. Select two consecutive vertices of the pentagon and construct a line (not a segment)
9. Repeat Step 8 for each remaining pairs of consecutive vertices.
10. You have now created a "star polygon" around the pentagon. If necessary, select point *B* and drag it so that the entire star is visible.




Now you will find the ratios of different segments. Using the figure above:

1. Select the endpoints of a segment on your diagram that corresponds to *c*. Measure the length of segment *c*. **Lengths will vary.**
2. Select the endpoints of a segment on your diagram that corresponds to *d*. Measure the length of segment *d*. **Lengths will vary**

3. Go to Measure and choose Calculate. Then, click on the length representing segment *c*, click on the division symbol, and then click on the length representing segment *d*. Click OK. Where have you seen this number before?

   **It is the Golden Ratio.**

4. Grab and drag Point *B*. What is changing? What is remaining constant?

   **The lengths are changing; the ratio remains constant.**

5. Select the endpoints of a segment on your diagram that corresponds to *e*. Measure the length of segment *e*. **Lengths will vary.**
6. Go to Measure and choose Calculate. Then, click on the length representing segment *e*, click on the division symbol, and then click on the length representing segment *c*. Click OK. There it is again!
7. Grab and drag Point *B*. What is changing? What is remaining constant?

   **The lengths are changing; the ratio remains constant.**

8. Select the vertices of the star polygon **in order**. Go to Construct and select Polygon Interior (this will shade the interior). Click on the interior of the star polygon so it is highlighted, go to Measure, and select Area.
9. Go to Display and choose Hide Polygon.
10. Select the vertices of the pentagon **in order**. Go to Construct and select Polygon Interior. Click on the interior of the pentagon so it is highlighted, go to Measure, and select Area.
11. Go to Measure and select Calculate. Click on the larger area, the division symbol, and then the smaller area. Click OK. How does this number compare with The Golden Ratio? Can you use the Calculate feature to justify your answer?
    **The number (3.236) is twice as big as the Golden Ratio. If the students divide the area of the "star polygon" by the area of the pentagon, they will get 2.**

12. Grab and drag Point *B*. What is changing? What is remaining constant?

    **The areas are changing; the ratio of 2 remains constant.**

## Fibonacci numbers and the Golden Ratio

In 1202, Leonardo Fibonacci proposed a problem with an answer that was in the form of the following sequence:

1, 1, 2, 3, 5, 8, 13…

This set of numbers is known as the Fibonacci sequence. As you may notice, starting with the third value, each successive value is the sum of the previous two values. What are the next two values in the sequence?
**21 and 34.**

You are now going to use an Excel spreadsheet to develop the Fibonacci sequence and look at the values of successive quotients.

1. Open an Excel spreadsheet. In cells A1 and A2, enter the number 1. With cell A3 highlighted, go to the formula bar and enter =SUM(A1:A2). Press enter. You should now see 2 in cell A3.
2. Select cell A3 and grab the square in the left hand corner of the cell (you will see a plus sign). Drag down to cell A20 and release. You now have the first twenty values of the Fibonacci sequence. **A quick check:** The value in cell A20 should be 6765.
3. Select cell B3. In the formula bar, enter =(A3/A2). Then, select cell B3 and drag this formula down to cell B20.
4. What do you notice about the values in column B?

   **They are approaching the Golden Ratio.**

| | |
|---|---|
| 1 | |
| 1 | |
| 2 | 2 |
| 3 | 1.5 |
| 5 | 1.666667 |
| 8 | 1.6 |
| 13 | 1.625 |
| 21 | 1.615385 |
| 34 | 1.619048 |
| 55 | 1.617647 |
| 89 | 1.618182 |
| 144 | 1.617978 |
| 233 | 1.618056 |
| 377 | 1.618026 |
| 610 | 1.618037 |
| 987 | 1.618033 |
| 1597 | 1.618034 |
| 2584 | 1.618034 |
| 4181 | 1.618034 |
| 6765 | 1.618034 |

**These are the values that should appear in the table.**

## Assessment

### Teacher's Guide

#### Introduction

The purpose of the assessment questions is to determine if the students mastered the objectives that were taught. Additional applications of the Golden Ratio, such as Golden Triangles and the appearance of the Golden Ratio in the Fibonacci Sequence are included in the assessment. No additional knowledge is required for these applications.

#### Objectives Covered

Students will be able to:

- Solve for the lengths of missing segments by using the Golden Ratio
- Identify the appearance of the Golden Ratio in additional contexts.
- Describe uses of the Golden Ratio in art and architecture.

#### Tools/Materials Needed for Assessment

Assessment sheet
Homework sheet

#### Administering the Assessment

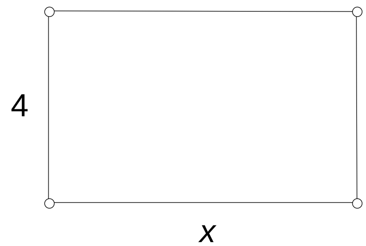
Teachers may use the assessment questions as test questions or quiz questions. All of the questions require students to use ratios or proportions to solve for missing segment lengths. Note that questions 1c and 6c on the assessment require students to use the Pythagorean Theorem in order to obtain the answer.

Most questions in the homework assignment reinforce the concepts introduced in the worksheets. One question in the homework assignment asks students to discover the appearance of the Golden Ratio in the ratios of successive terms of the Fibonacci sequence. Because students may not be familiar with the Fibonacci sequence, it is defined in the question.

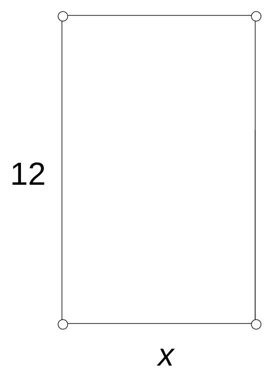
**Golden Ratio Assessment Questions** **NAME________________________**

1) Find the length of the indicated segment of each Golden Rectangle.
Round your answers to the nearest hundredth.

a) *x* = ______

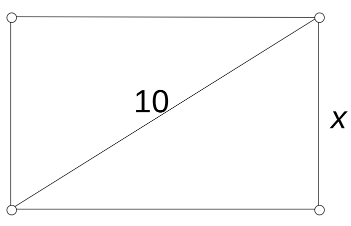



b) *x* = ______

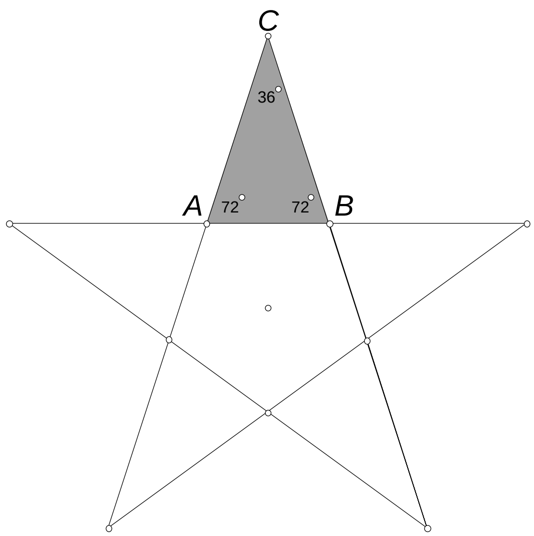



c) *x* = ______

*Hint*: Consider the right triangle formed by two adjacent sides of the rectangle and the diagonal. The length of the shorter leg of this triangle is *x*. By the Golden Ratio, the length of the long leg must be 1.618*x*. Now use the Pythagorean Theorem to solve for *x*.



2) The five isosceles triangles formed around a star are known as Golden Triangles. One such Golden Triangle is shaded in the figure below. In each Golden Triangle, the vertex angle has a measure of $36^{o}$ and the base angles each have a measure of $72^{o}$. The ratio of the length of a leg to the length of the base is equal to the Golden Ratio. Use the figure below to answer the following questions.




a) If the length of segment *AB* is 13, find the length of segment *AC*. *AC* = ________

b) If the length of segment *BC* is 25, find the length of segment *AB*. *AB* = ________

c) If the length of segment *AC* is 5.7, find the length of segment *BC*. *BC* = ________

3) Which of the following dimensions conform most closely to the definition of a Golden Rectangle? Explain.

Rectangle A: length = 26, width = 13

Rectangle B: length = 8, width = 5

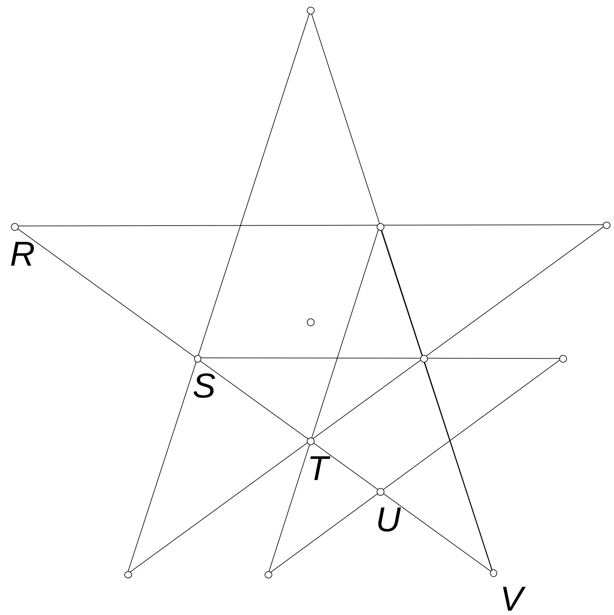
Rectangle C: length = 100, width = 80

______________________________________________________________________________

______________________________________________________________________________

4) In the figure below, the labeled segments have the following lengths.

$TV = SU = RS = 1$ and $UV = ST = 0.618$.




a) Find the ratio of *TV* to *UV*. ________

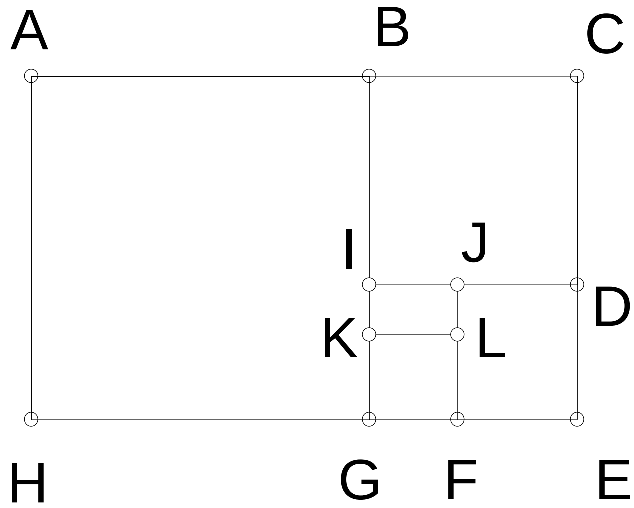
b) Find the ratio of *UV* to *TU*. ________

c) Find the ratio of *RV* to *SV*. ________

d) How do the answers to a), b), and c) compare? ________________________

e) What is the significance of the value of these ratios? ________________________

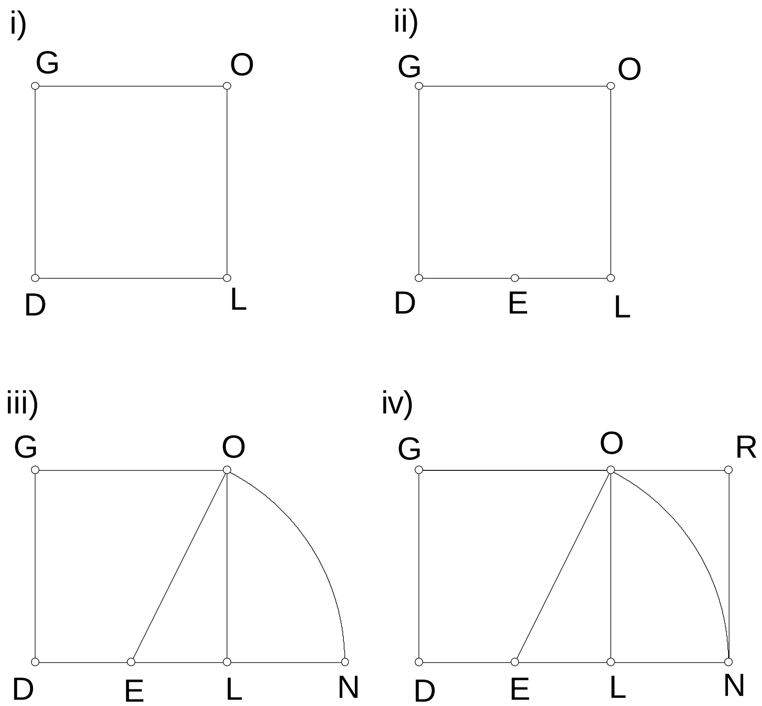
5) Each of the rectangles in the figure below is a Golden Rectangle. The lengths and widths in the table are rounded off to the nearest hundredth. Complete the table.




| Rectangle | ACEH | BCEG | IDEG | IJFG | IJLK |
|---|---|---|---|---|---|
| Length | 6.472 | 4.000 | 2.472 | 1.528 | 0.944 |
| Width | 4.000 | | | | |

6) Consider the construction of the Golden Rectangle (Rectangle GRND) which starts with square GOLD. Point E is constructed as a midpoint of segment DL and arc ON is constructed as an arc of the circle with its center at E. The steps in this construction are shown below.

Find the value of the following lengths if DL=2. Round to the nearest thousandth.




a) $DG =$ ______.

b) $EL =$ ______.

c) $EO =$ ______.

d) $EN =$ ______.

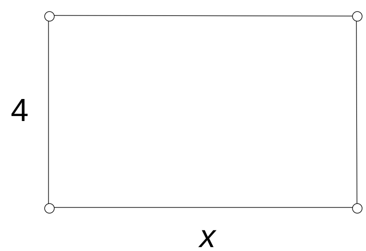
e) $DN =$ ______.

f) The value of the Golden Ratio $\frac{DN}{DG}$ is _________.

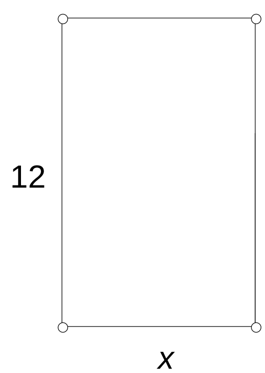
**Golden Ratio Assessment Questions: Answer Key**

1) Find the length of the indicated segment of each Golden Rectangle.
Round your answers to the nearest hundredth.

a) $x$ = **6.47**

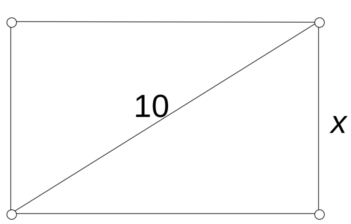



b) $x$ = **7.42**

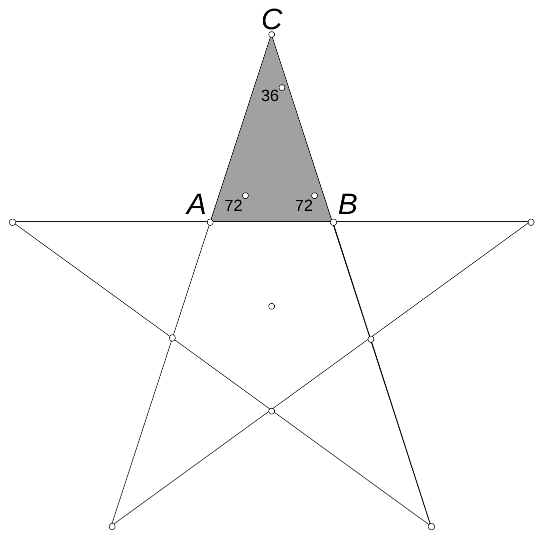



c) $x$ = **5.26**

*Hint*: Consider the right triangle formed by two adjacent sides of the rectangle and the diagonal. The length of the shorter leg of this triangle is $x$. By the Golden Ratio, the length of the long leg must be $1.618x$. Now use the Pythagorean Theorem to solve for $x$.



2) The five isosceles triangles formed around a star are known as Golden Triangles.  One such Golden Triangle is shaded in the figure below.  In each Golden Triangle, the vertex angle has a measure of $36^{o}$ and the base angles each have a measure of $72^{o}$.  The ratio of the length of a leg to the length of the base is equal to the Golden Ratio.  Use the figure below to answer the following questions.




a) If  the length of segment *AB* is 13, find the length of segment *AC*. *AC* = **21.0**

b) If the length of segment *BC* is 25, find the length of segment *AB*. *AB* = **15.4**

c) If the length of segment *AC* is 5.7, find the length of segment *BC*. *BC* = **5.7**

3) Which of the following dimensions conform most closely to the definition of a Golden Rectangle?  Explain.

Rectangle A: length = 26, width = 13

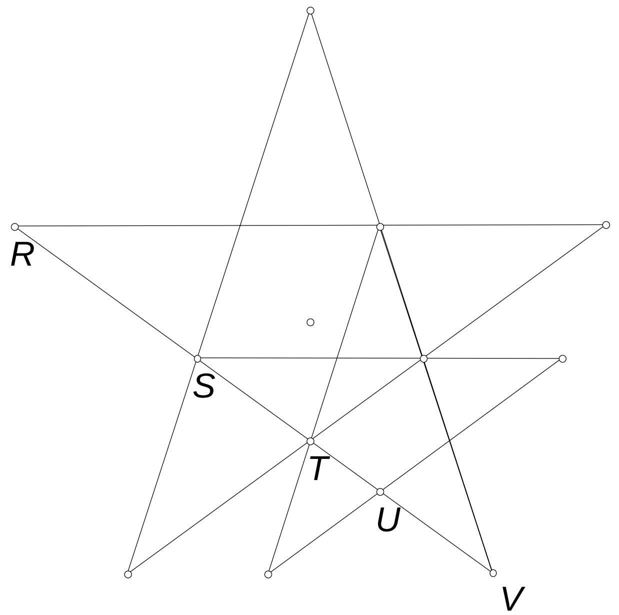
Rectangle B: length = 8, width = 5

Rectangle C: length = 100, width = 80

**Rectangle B.  The ratio of its length to its width is 8/5 = 1.6, which is approximately equal to the Golden Ratio.**

4) In the figure below, the labeled segments have the following lengths.

$TV = SU = RS = 1$ and $UV = ST = 0.618$.




a) Find the ratio of *TV* to *UV*. **<u>1.618</u>**

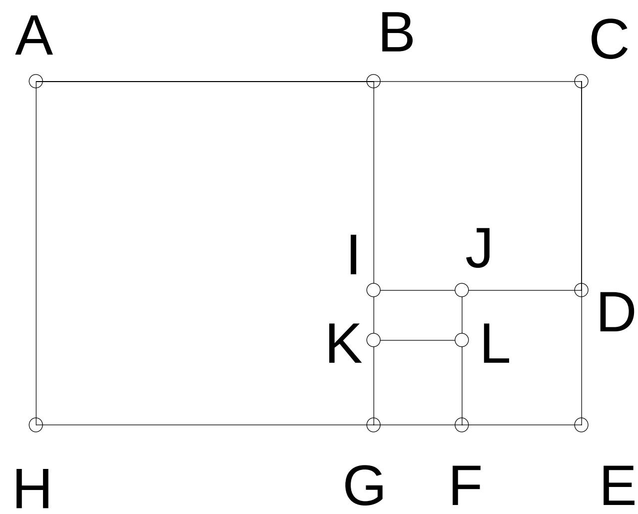
b) Find the ratio of *UV* to *TU*. **<u>1.618</u>**

c) Find the ratio of *RV* to *SV*. **<u>1.618</u>**

d) How do the answers to a), b), and c) compare? **<u>The answers are equal.</u>**

e) What is the significance of the value of these ratios? **<u>They are the Golden Ratio.</u>**

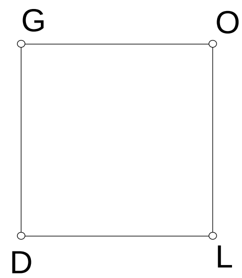
5) Each of the rectangles in the figure below is a Golden Rectangle.  The lengths and widths in the table are rounded off to the nearest hundredth.  Complete the table.



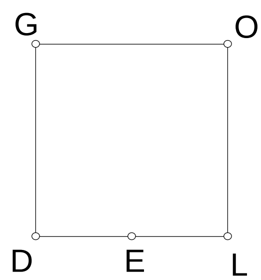

| **Rectangle** | **ACEH** | **BCEG** | **IDEG** | **IJFG** | **IJLK** |
|---|---|---|---|---|---|
| Length | 6.472 | 4.000 | 2.472 | 1.528 | 0.944 |
| Width | 4.000 | **2.472** | **1.528** | **0.944** | **0.584** |

6) Consider the construction of the Golden Rectangle (Rectangle GRND) which starts with square GOLD. Point E is constructed as a midpoint of segment DL and arc ON is constructed as an arc of the circle with its center at E. The steps in this construction are shown below.

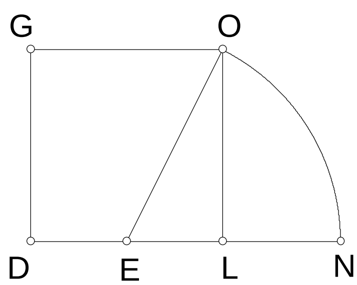
i)



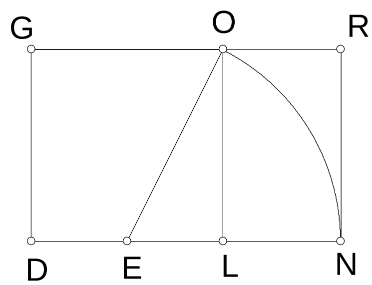

ii)




iii)




iv)




a) $DG$ =**2**.

b) $EL$ = **1**.

c) $EO$ = **2.236.**

d) $EN$ = **2.236**.

e) $DN$ = **3.236**.

f) Then the value of the Golden Ratio $\frac{DN}{DG}$ is **1.618**.